# Characterization of Linear Independence of two vectors in Euclidean Spaces using Triangle Inequality

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**Abstract**

This paper discusses a result that has been used mainly in Convex Analysis with connection to the theory of Linear and Non-Linear Programming Problems, which is a necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ in the Euclidean Space $\mathbb{R}^n$ to be linearly independent, which involves the usual definition of a norm in the said space. It has been used directly in works like those of Bazaraa [1], Hadley [2], Mittal [6], among others as a result which follows trivially by using some intuitive notions. This paper is an attempt to give a formal proof to the said result.

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# 1 Introduction and Related Definitions

## 1.1 $\mathbb{R}^n$ as a Vector Space

The set $\mathbb{R}^n$ ($n \in \mathbb{N}$) defined as

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

which is the set of all n-tuples of real numbers, forms a vector space over the field $(\mathbb{R}, +, \cdot)$ under the following operations.

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$\alpha \cdot (x_1, x_2, \ldots, x_n) = (\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_n)$$

for all $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ and every $\alpha$ in $\mathbb{R}$. Here, the sign $+$ between two n-tuples denotes what we call as **vector addition**, and the same sign between the components denote the addition operation on real numbers. In the same way, the symbol $\cdot$ between the real number $\alpha$ and the n-tuple denotes what we call as **scalar multiplication** and the same sign which appear between the components denote the multiplication of real numbers. This sign may be replaced by a mere juxtaposition, with no alteration in the meaning and contexts. So, the same symbols carry different meanings in different contexts.

These terminologies are according to the standards of Linear Algebra. For a verification of this being a vector space, please refer to [5] or any standard textbook on Linear Algebra.

Since no confusion shall arise, we shall call the elements of $\mathbb{R}^n$ as **vectors** (which are denoted by symbols with arrowheads) and the real numbers as **scalars**, throughout this paper. We may conveniently and equivalently replace the n-tuples by row or column matrices of real numbers, if necessary.

## 1.2 Dot Product and Norms

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ be any two vectors. Then their **dot product**, also called **scalar product**, is defined as

$$\vec{x} \cdot \vec{y} = (x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^{n} x_k y_k.$$

The scalar product defined in this manner enjoys the following properties. The proofs are straightforward and follow immediately from the definition.

- For every pair of vectors $\vec{x}$ and $\vec{y}$, $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$. This is called **Commutative property**.
- For every triplet of vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$, $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$. This is called **Distributive property**. A similar property can be stated for the sum of vectors being to the left of the dot in the scalar product, but it is redundant due to the commutative property stated above.
- For all scalars $\alpha$, $(\alpha\vec{x}) \cdot \vec{y} = \vec{x} \cdot (\alpha\vec{y}) = \alpha(\vec{x} \cdot \vec{y})$. This is called **Associative property**.

For convenience, we denote $\vec{x} \cdot \vec{x}$ by $(\vec{x})^2$. This shall carry no other meaning other than what we have just stated. We shall not define the vectors with any other numerical exponents. Note that the definition assures $(\vec{x})^2 \geq 0$.

By repeated application of the distributive property, we may verify the following results, quite analogous to the ones in the context of scalars.

- For all vectors $\vec{x}$ and $\vec{y}$, we have $(\vec{x}+\vec{y})^2 = (\vec{x})^2 + (\vec{y})^2 + 2\,\vec{x}\cdot\vec{y}$.
- For all vectors $\vec{x}$ and $\vec{y}$, we have $(\vec{x}-\vec{y})^2 = (\vec{x})^2 + (\vec{y})^2 - 2\,\vec{x}\cdot\vec{y}$.
- For all vectors $\vec{x}$ and $\vec{y}$, we have $(\vec{x}+\vec{y})\cdot(\vec{x}-\vec{y}) = (\vec{x})^2 - (\vec{y})^2$.

We will use the first two results in to establish the main result this paper concerns.

Now, we define the norm of a vector $\vec{x}$. The **norm**, also called the **length**, of a vector $\vec{x}$ is defined as

$$\|\vec{x}\| = \sqrt{(\vec{x})^2} = \sqrt{\vec{x}\cdot\vec{x}} = \left(\sum_{k=1}^{n} {x_k}^2\right)^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

where, $\vec{x} = (x_1, x_2, \ldots, x_n)$. This norm enjoys the following properties. For the proofs, see [5]. The proof can also be found in some elementary textbooks on Functional Analysis.

- For all vectors $\vec{x}$, $\|\vec{x}\| \geq 0$.
- $\vec{x} = \vec{0} \Leftrightarrow \|\vec{x}\| = 0$, where, $\vec{0} = (0, 0, \ldots, 0)$, an n-tuple of zeroes.
- For all scalars $\alpha$ and vectors $\vec{x}$, $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$.
- For all vectors $\vec{x}$ and $\vec{y}$, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

We define a vector with norm 1 as a **unit vector**.

The final property stated above is what we refer to as **Triangle Inequality**. This is what we shall use to provide a characterization to two linearly independent vectors in $\mathbb{R}^n$.

Now, we shall be giving the definition of linear dependence and independence of two vectors. This definition can be extended to any finite number of vectors, but we shall not do so in this paper. For the extension, see [2], [4] and [5].

### 1.3 Linear Dependence and Independence

Two vectors $\vec{x}$ and $\vec{y}$ are said to be **linearly dependent** if, either one (or both) of them are zero vectors or if we have a scalar $\lambda \neq 0$ such that $\vec{y} = \lambda\vec{x}$, whenever none of the vectors are zero. If two vectors are not linearly dependent, they are said to be **linearly independent**.

## 2 The Characterization Theorem and its Corollary

**Theorem 1** (Characterization of Linear Independence)**.** *Two vectors $\vec{a}$ and $\vec{b}$ are linearly independent if and only if $\|\vec{a}+\vec{b}\| < \|\vec{a}\| + \|\vec{b}\|$.*

*Proof.* To establish the result, we make the following claim.

**Claim** : $|\vec{a}\cdot\vec{b}| = \|\vec{a}\|\ \|\vec{b}\|$ holds if and only if the two vectors involved are linearly dependent.

Let the vectors be linearly dependent. If at least one of them are zero vectors, then we know that their norm is zero, giving us the equality. If none of them are zero vectors, then there is some scalar $\lambda$ such that $\vec{b} = \lambda\vec{a}$. Now, we have, using the properties of norm and scalar product,

$$|\vec{a} \cdot \vec{b}| = |\vec{a} \cdot (\lambda\vec{a})| = |\lambda| \ \|\vec{a}\|^2$$

and,

$$\|\vec{a}\| \ \|\vec{b}\| = \|\vec{a}\| \ (|\lambda| \ \|\vec{a}\|) = |\lambda| \ \|\vec{a}\|^2.$$

This means, the equality in the claim holds.

Conversely, assume that the equality holds. If any one of the vectors are zero, then they are linearly dependent, by definition. If none of them are zero, then using associative property of scalar product, the above equality gives

$$\frac{1}{\|\vec{a}\| \ \|\vec{b}\|}|\vec{a} \cdot \vec{b}| = 1 \Rightarrow \left| \frac{\vec{a}}{\|\vec{a}\|} \cdot \frac{\vec{b}}{\|\vec{b}\|} \right| = 1.$$

Let us denote $\dfrac{\vec{a}}{\|\vec{a}\|}$ by $\vec{x}$ and $\dfrac{\vec{b}}{\|\vec{b}\|}$ by $\vec{y}$. Then, $\vec{x}$ and $\vec{y}$ are unit vectors. From the above equality, we get that $|\vec{x} \cdot \vec{y}| = 1$. So we can have following two cases.

**Case 1** : $\vec{x} \cdot \vec{y} = 1$

In this case, we have, by definition of a norm, we have,

$$\|\vec{x} - \vec{y}\|^2 = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) = (\vec{x})^2 + (\vec{y})^2 - 2\,\vec{x} \cdot \vec{y}.$$

As $\vec{x} \cdot \vec{y} = 1$ and both the vectors are unit vectors, the right hand side of the above equality becomes $1 + 1 - 2 \times 1 = 0$, giving us $\|\vec{x} - \vec{y}\| = 0$ and hence, $\vec{x} = \vec{y}$. Replacing these notations by actual vectors, we have obtained, $\dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{\vec{b}}{\|\vec{b}\|}$, which means, $\vec{b} = \lambda\vec{a}$, where $\lambda = \dfrac{\|\vec{b}\|}{\|\vec{a}\|}$. As $\lambda \neq 0$, it follows that the vectors are linearly dependent.

**Case 2** : $\vec{x} \cdot \vec{y} = -1$

In this case, we have, by definition of a norm, we have,

$$\|\vec{x} + \vec{y}\|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = (\vec{x})^2 + (\vec{y})^2 + 2\,\vec{x} \cdot \vec{y}.$$

As $\vec{x} \cdot \vec{y} = -1$ and both the vectors are unit vectors, the right hand side of the above equality becomes $1 + 1 + 2 \times (-1) = 0$, giving us $\|\vec{x} + \vec{y}\| = 0$ and hence, $\vec{y} = -\vec{x}$. Replacing these notations by actual vectors, we have obtained, $\dfrac{\vec{b}}{\|\vec{b}\|} = -\dfrac{\vec{a}}{\|\vec{a}\|}$, which means, $\vec{b} = \lambda\vec{a}$, where $\lambda = -\dfrac{\|\vec{b}\|}{\|\vec{a}\|}$. As $\lambda \neq 0$, it follows that the vectors are linearly dependent.

This shows the linear dependence of the two vectors which in the equality stated in our claim. Since our claim was correct, we may use it to establish the characterization theorem, as shown below.

Assume that $\vec{a}$ and $\vec{b}$ are linearly independent. To show that the strict inequality holds, assume the contrary. Then, from triangle inequality, we must have,

$$\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$$

which, on squaring, gives,

$$\|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\vec{a}\cdot\vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\|\vec{a}\|\,\|\vec{b}\|$$

which on cancellation gives,

$$|\vec{a}\cdot\vec{b}| = \|\vec{a}\|\ \|\vec{b}\|.$$

This means that the vectors are linearly dependent, as per our claim. Hence, combining this with our assumption of linear independence, we get a contradiction. Hence, we must have

$$\|\vec{a}+\vec{b}\| < \|\vec{a}\| + \|\vec{b}\|.$$

Conversely, assume that $\|\vec{a}+\vec{b}\| < \|\vec{a}\| + \|\vec{b}\|$. To show the linear independence, assume the contrary, that is, assume that the vectors are linearly dependent. Now, by our claim, we get that

$$|\vec{a}\cdot\vec{b}| = \|\vec{a}\|\ \|\vec{b}\|$$

which gives

$$\|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\vec{a}\cdot\vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\|\vec{a}\|\,\|\vec{b}\|.$$

Using definition of norm and scalar product, the last equality gives

$$\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$$

after we take the square roots, since both the quantities in the equality are non-negative. But, the last equality, when combined with our initial assumption, gives a contradiction. Hence, the vectors must be linearly independent. □

The above theorem is the central concern of this paper. As a corollary, we give an analogous characterization to linear dependence, which has been stated and proved as Theorem 2 in this paper.

**Theorem 2** (Characterization of Linear Dependence)**.** *Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$.*

*Proof.* If we assume the dependence, then we will get the equality. This is because if we assume otherwise, the triangle inequality gives, $\|\vec{a}+\vec{b}\| < \|\vec{a}\| + \|\vec{b}\|$ which gives independence (as per Theorem 1), which is a contradiction. Conversely, if we assume that the equality holds, then we must get the dependence. This is because, if we assume the independence, Theorem 1 gives $\|\vec{a}+\vec{b}\| < \|\vec{a}\| + \|\vec{b}\|$, which contradicts our assumption of equality. □

# 3 A Consequence - The Extension of Geometrical Notion of a vector from $\mathbb{R}^2$ to any Euclidean Space

The major consequence of these theorems is that we can extend the definition of angle between two vectors that we have introduced in $\mathbb{R}^2$, in any Euclidean space. We know, from Cauchy-Schwarz inequality, that $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \ \|\vec{b}\|$ (for a variety of proofs, see [4]). This means

$$-1 \leq \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \ \|\vec{b}\|} \leq 1.$$

So, if $\theta$ is any real number, for any two non-zero vectors $\vec{a}$ and $\vec{b}$, the expression

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \ \|\vec{b}\|}$$

is well-defined. Let us take $\theta \in [0, \pi]$. Then we define $\theta$ to be the angle between the two vectors involved. Using this definition, as in $\mathbb{R}^2$, we get the linear dependence of the vectors iff $\theta = 0$ or $\theta = \pi$. Also, if we define, the condition of perpendicularity of these two vectors to be $\vec{a} \cdot \vec{b} = 0$, then, $\theta = \dfrac{\pi}{2}$ is a necessary and sufficient condition for perpendicularity. Hence, these theorems help us to extend the geometrical notion of a vector in $\mathbb{R}^2$ to any Euclidean Space. These concepts have been used in works like [2], however, the notion was purely intuitive. This paper, now, has given a much more logical explanation to this intuitive idea.

# References

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